Optimality

www.opt.reapress.com

Opt. Vol. 2, No. 2 (2025) 78–83.

Paper Type: Original Article

Non-Extendable Rational Diophantine 3-tuples Comprising Polynomials and Special Figurate Numbers

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Citation:

Received: 11 July 2024 *Revised: 10 September 2024* *Accepted: 19 January 2025*	Sangeetha, V., Anupreethi, T., & Manju Somanath. (2025). Non-extendable rational diophantine 3-tuples comprising polynomials and special figurate numbers. *Optimality*, 2(2), 78-83.

Abstract

A rational Diophantine 3-tuple is a family of three non-zero rationals x, y, z with the property $D(n)$ such that $xy + n, yz + n, xz + n$ are perfect squares. In this paper, a pair of rational polynomials and some special figurate numbers are scrutinized for their extension as triples and are proved to be non-extendable as quadruples.

Keywords: Diophantine triples, Diophantine quadruples, Regular rational diophantine triples, Irregular rational diophantine quadruples.

1|Introduction

The necessary literature survey that paved us a way to write this paper is given as follows: A collection $\{x_1, x_2 \cdots x_m\}$ of m distinct non-zero rationals is called a rational Diophantine m-tuple if $x_i x_j + 1$ is a perfect square for all $1 \leq i < j \leq m$ [1].The greek Mathematician Diophantus of Alexandria first analysed the problem of searching four numbers such that the product of any two of them increased by unity is a perfect square.He spotted a set of four positive rationals $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ with this property.Followed by Diophantus many Mathematicians [2–4] searched the triples or quadruples that can be extended to the next consecutive tuples.Recently Gibbs found the first rational Diophantine sextuple $\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{48}, \frac{180873}{16}\right\}$. Meanwhile Dujella, Kazalicki, Mikic and Szikszai proved that there are infinitely many rational Diophantine sextuples.Though these legends discovered the Diophantine m-tuples that are extendible,there are some cases where the rational Diophantine triples cannot be extended to

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https://doi.org/10.22105/opt.v2i2.77

a rational Diophantine quadruple [5, 6]. In this paper we have focused on such kind of numbers say rational polynomials,polygonal numbers,
Centered pyramidal numbers and so on.We prove their non-extendability by showing that any two numbers of those quadruples does not meet the requirement of Diophantine m-tuples for the particular property. To acquire deep knowledge in Diophantine m-tuples and their generalizations, the readers are diverted to [7].

Definition 1. *A set of m nonzero rationals* $\{a_1, a_2 \cdots a_m\}$ *is called a Regular Diophantine m-tuple if* $a_i a_j + q$ *is a perfect square for all* $i, j = 1, 2, \cdots m$*.If the condition is not satisfied for any one pair of* (a_i, a_j)*, then it is known as an irregular Diophantine m-tuple.*

2|Regular Diophantine triples and irregular Diophantine quadruples with respect to Rational Numbers

Problem 1: Consider the 2-tuple $(R_{Q1}, R_{Q2}) = \left(\frac{1}{4W}, \frac{1}{n^2W}\right)$ satisfying the property $D(n^2W^2 + 1)\ \forall\ n \geq 3$. To search for its extendability, let R_{Q3} be any other rational number with the same property. Here we can see that

$$R_{Q1}.R_{Q3} + (n^2W^2 + 1) = \psi^2 \tag{1}$$

$$R_{Q2}.R_{Q3} + (n^2W^2 + 1) = \Omega^2 \tag{2}$$

Eliminating R_{Q3} between (1) and (2), we get

$$(n^2W^2 + 1)(R_{Q2} - R_{Q1}) = \psi^2 R_{Q2} - \Omega^2 R_{Q1} \tag{3}$$

The choice

$$\psi = j + R_{Q1}H; \Omega = j + R_{Q2}H \tag{4}$$

Substitute equation (4) in equation (3) which yields the succeeding equation $(n^2W^2 + 1) = j^2 - R_{Q1}.R_{Q2}H^2$. For the choice $H = 1$,the values of j, ψ are found to be $j = \frac{1 + 2n^2W^2}{2nW}$, $\psi = \frac{4n^2W^2 + n + 2}{4nW}$. The values of ψ can be employed to derive the rational number $R_{Q3} = \frac{(n + 2)^2 + W^2(8n^3)}{4n^2W}$. Hence (R_{Q1}, R_{Q2}, R_{Q3}) is a Regular rational Diophantine triple with property $D(n^2W^2 + 1), \forall n \geq 3$.

Theorem 0.1. *Let* $R_{Q1} = \frac{1}{4W}, R_{Q2} = \frac{1}{n^2W}, R_{Q3} = \frac{(n + 2)^2 + W^2(8n^3)}{4n^2W}$ *and* $R_{Q4} = \frac{(n + 4)^2 + 16n^3W^2}{4n^2W}$*.Then* $(R_{Q1}, R_{Q2}, R_{Q3}, R_{Q4})$ *is an Irregular rational Diophantine quadruple with the property* $D(n^2W^2 + 1)$ *,*$\forall\ n \geq 3$*.*

Proof:

$$R_{Q1}R_{Q2} + (n^2W^2 + 1) = \left(\frac{1}{4W}\right)\left(\frac{1}{n^2W}\right) + (n^2W^2 + 1) = \left(\frac{1 + 2n^2W^2}{2nW}\right)^2$$

$$R_{Q1}R_{Q3} + (n^2W^2 + 1) = \left(\frac{(n + 2) + 4n^2W^2}{4nW}\right)^2$$

$$R_{Q1}R_{Q4} + (n^2W^2 + 1) = \frac{(n + 4)^2 + (16n^3)W^2 + 16n^4W^4 + 16n^2W^2}{16n^2W^2}$$

$$R_{Q2}R_{Q3} + (n^2W^2 + 1) = \left(\frac{(n + 2) + 2n^3W^2}{2n^2W}\right)^2$$

$$R_{Q2}R_{Q4}+(n^2W^2+1)=\left(\frac{1}{n^2W}\right)\left(\frac{(n+4)^2+16n^3W^2}{4n^2W}\right)+(n^2W^2+1)$$
$$=\left(\frac{(n+4)+2n^3W^2}{2n^2W}\right)^2$$
$$R_{Q3}R_{Q4}+(n^2W^2+1)=\left(\frac{(n+2)^2+W^2(8n^3)}{4n^2W}\right)\left(\frac{(n+4)^2+16n^3W^2}{4n^2W}\right)$$
$$+(n^2W^2+1)$$
$$=\left(\frac{(n^2+6n+8)+12n^3W^2}{4n^2W}\right)^2$$

For all the combinations of the triples satisfying $D(n^2W^2+1)$,$\forall\ n\geq 3$, except
$R_{Q1}R_{Q4}+(n^2W^2+1)$ are perfect Squares. Hence $(R_{Q1},R_{Q2},R_{Q3},R_{Q4})$ is an Irregular rational Diophantine quadruple.

Corollary 0.1. *Let* $R_{Q1}=\dfrac{1}{4W},R_{Q2}=\dfrac{1}{n^2W},R_{Q3}=\dfrac{(n+2)^2+W^2(8n^3)}{4n^2W}$ *and* $R_{Q4}=\dfrac{(n+4)^2+16n^3W^2}{4n^2W}$. *Then* $(R_{Q1},R_{Q2},R_{Q3},R_{Q4})$ *is a* $D(n^2W^2-1)\ \forall\ n\geq 3$, *Irregular rational Diophantine quadruple.*

Example:

TABLE 1. Irregular rational Diophantine quadruple.

$n\geq 3$	$(R_{Q1},R_{Q2},R_{Q3},R_{Q4})$	$D(n^2W^2+1)$
3	$\left(\frac{1}{4W},\frac{1}{3^2W},\frac{216W^2+5^2}{6^2W},\frac{432W^2+7^2}{6^2W}\right)$	$D(3^2W^2+1)$
4	$\left(\frac{1}{4W},\frac{1}{4^2W},\frac{128W^2+3^2}{4^2W},\frac{16W^2+1}{W}\right)$	$D(4^2W^2+1)$
5	$\left(\frac{1}{4W},\frac{1}{5^2W},\frac{1000W^2+7^2}{10^2W},\frac{2000W^2+9^2}{10^2W}\right)$	$D(5^2W^2+1)$

Problem 2: Examine the 2-tuple $(R_{Q5},R_{Q6})=\left(\frac{1}{W_1},\frac{1}{4^{n_1}W_1}\right)$ fulfilling the property $D(4^{n_1}W_1^2+2)$.To determine whether it can be extended, let R_{Q7} be a
different rational number, achieving the same property. Here we can see that

$$R_{Q5}R_{Q7}+(4^{n_1}W_1^2+2)=\psi^2 \tag{5}$$
$$R_{Q6}R_{Q7}+(4^{n_1}W_1^2+2)=\Omega^2 \tag{6}$$

Applying the same procedure described in the previous problem, it is obtained the Regular rational Diophantine triple, $\left(\frac{1}{W_1},\frac{1}{4^{n_1}W_1},\frac{(1+2^{n_1})^2+2(8^{n_1})W_1^2}{4^{n_1}W_1}\right)$
satisfying $D(4^{n_1}W_1^2+2)$.

Theorem 0.2. *Let* $R_{Q5}=\dfrac{1}{W_1},R_{Q6}=\dfrac{1}{4^{n_1}W_1},R_{Q7}=\dfrac{(1+2^{n_1})^2+2(8^{n_1})W_1^2}{4^{n_1}W_1}$
and $R_{Q8}=\dfrac{(2+2^{n_1})^2+4(8^{n_1})W_1^2}{4^{n_1}W_1}$. *Then* $(R_{Q5},R_{Q6},R_{Q7},R_{Q8})$*is a*
$D(4^{n_1}W_1^2+2)$ *Irregular rational Diophantine quadruple.*

Proof:

$$R_{Q5}R_{Q6}+(4^{n_1}W_1^2+2)=\left(\frac{1}{W_1}\right)\left(\frac{1}{4^{n_1}W_1}\right)+(4^{n_1}W_1^2+2)$$
$$=\left(\frac{1+4^{n_1}W_1^2}{2^{n_1}W_1}\right)^2$$
$$=\left(\frac{(1+2^{n_1})+4^{n_1}W_1^2}{2^{n_1}W_1}\right)^2$$
$$R_{Q5}R_{Q7}+(4^{n_1}W_1^2+2)=\left(\frac{1}{W_1}\right)\left(\frac{(1+2^{n_1})^2+2(8^{n_1})W_1^2}{4^{n_1}W_1}\right)+(4^{n_1}W_1^2+2)$$
$$=\left(\frac{(1+2^{n_1})+4^{n_1}W_1^2}{2^{n_1}W_1}\right)^2$$
$$R_{Q5}R_{Q8}+(4^{n_1}W_1^2+2)=\frac{(2+2^{n_1})^2+(4^{n_1}W_1^2)^2+2(4^{n_1})W_1^2(2(2^{n_1})+1)}{(2^{n_1}W_1)^2}$$
$$R_{Q6}R_{Q7}+(4^{n_1}W_1^2+2)=\left(\frac{1}{4^{n_1}W_1}\right)\left(\frac{(1+2^{n_1})^2+2(8^{n_1})W_1^2}{4^{n_1}W_1}\right)+(4^{n_1}W_1^2+2)$$
$$=\left(\frac{(1+2^{n_1})+8^{n_1}W_1^2}{4^{n_1}W_1}\right)^2$$
$$R_{Q6}R_{Q8}+(4^{n_1}W_1^2+2)=\left(\frac{1}{4^{n_1}W_1}\right)\left(\frac{(2+2^{n_1})^2+4(8^{n_1})W_1^2}{4^{n_1}W_1}\right)+(4^{n_1}W_1^2+2)$$

$$R_{Q6}R_{Q8}+(4^{n_1}W_1^2+2)=\left(\frac{(2+2^{n_1})+8^{n_1}W_1^2}{4^{n_1}W_1}\right)^2$$
$$R_{Q7}R_{Q8}+(4^{n_1}W_1^2+2)=\left(\frac{(1+2^{n_1})^2+2(8^{n_1})W_1^2}{4^{n_1}W_1}\right)\left(\frac{(2+2^{n_1})^2+4(8^{n_1})W_1^2}{4^{n_1}W_1}\right)$$
$$+(4^{n_1}W_1^2+2)$$
$$=\left(\frac{(2+3(2^{n_1})+4^{n_1})+3(8^{n_1})W_1^2}{4^{n_1}W_1}\right)^2$$

Since $R_{Q5}.R_{Q8}+(4^{n_1}W_1^2+2)$ is not perfect square, $(R_{Q5},R_{Q6},R_{Q7},R_{Q8})$ is an irregular rational Diophantine quadruple.

Corollary 0.2. *Let* $R_{Q5}=\dfrac{1}{W_1},R_{Q6}=\dfrac{1}{4^{n_1}W_1},R_{Q7}=\dfrac{(1+2^{n_1})^2+2(8^{n_1})W_1^2}{4^{n_1}W_1}$ *and* $R_{Q8}=\dfrac{(2+2^{n_1})^2+4(8^{n_1})W_1^2}{4^{n_1}W_1}$*.Then* $(R_{Q5},R_{Q6},R_{Q7},R_{Q8})$*is a* $D(4^{n_1}W_1^2-2)$ *Irregular rational Diophantine quadruple.*

3|Regular rational Diophantine triples and irregular rational Diophantine quadruples with respect to the general formula for the Polygonal Numbers

Problem:

Consider the 2-tuple $\left(G_{Pn_1}=\dfrac{(2W+1)n^2-(2W-1)n}{2},G_{Pn_2}=\dfrac{(3W+1)n^2-(3W-1)n}{2}\right)$ rewarding the property$D\left(\dfrac{n^4(3W^2-5W-1)+n^3(12W^2-2)-n^2(6W^2-5W+1)}{4}\right)$.To ascertain whether it can be extended,let G_{Pn_3} be a rational number such that

$$G_{Pn_1}G_{Pn_3}+\frac{n^4(3W^2-5W-1)+n^3(12W^2-2)-n^2(6W^2-5W+1)}{4}=\theta^2 \quad (7)$$

$$G_{Pn_2}G_{Pn_3}+\frac{n^4(3W^2-5W-1)+n^3(12W^2-2)-n^2(6W^2-5W+1)}{4}=\tau^2 \quad (8)$$

Eliminating G_{Pn_3} between (7) and (8)we get

$$\left(\frac{n^4(3W^2-5W-1)+n^3(12W^2-2)-n^2(6W^2-5W+1)}{4}\right)(G_{Pn_2}-G_{Pn_1})$$
$$=\theta^2G_{Pn_2}-\tau^2G_{Pn_1} \quad (9)$$

Consider

$$\theta=j+G_{Pn_1}H;\tau=j+G_{Pn_2}H \quad (10)$$

The application of the equation (11) in equation(9),we get new equation
$\left(\frac{n^4(3W^2-5W-1)+n^3(12W^2-2)-n^2(6W^2-5W+1)}{4}\right)=j^2-G_{Pn_1}G_{Pn_2}H^2$

Consider $H=1$.We get $j=\frac{3Wn^2}{2}$, deputy j and H in $\theta,\theta=\frac{(5W+1)n^2-(2W-1)n}{2}$
The rational number can be determined using the value of θ
$G_{Pn_3}=\frac{n^4(22W^2+15W+2)-n^3(32W^2-6W-4)+n^2(10W^2-9W+2)}{2[(2W+1)n^2-(2W-1)n]}$
Hence($G_{Pn_1},G_{Pn_2},G_{Pn_3}$) is a Regular rational Diophantine triple with the Property
$D\left(\frac{n^4(3W^2-5W-1)+n^3(12W^2-2)-n^2(6W^2-5W+1)}{4}\right)$

Theorem 0.3. *Let* $G_{Pn_1}=\frac{(2W+1)n^2-(2W-1)n}{2}$,
$G_{Pn_2}=\frac{(3W+1)n^2-(3W-1)n}{2}$,
$G_{Pn_3}=\frac{n^4(22W^2+15W+2)-n^3(32W^2-6W-4)+n^2(10W^2-9W+2)}{2[(2W+1)n^2-(2W-1)n]}$ *and*
$G_{Pn_4}=\frac{n^4(78W^2+41W+5)+n^3(-120W^2+12W+10)+n^2(42W^2-29W+5)}{2[(3W+1)n^2-(3W-1)n]}$.
Then $(G_{Pn_1},G_{Pn_2},G_{Pn_3},G_{Pn_4})$ *is a*
$D\left(\frac{n^4(3W^2-5W-1)+n^3(12W^2-2)-n^2(6W^2-5W+1)}{4}\right)$ *irregular*
rational Diophantine quadruple.

Proof: For all the combinations of the quadruple satisfying
$D\left(\frac{n^4(3W^2-5W-1)+n^3(12W^2-2)-n^2(6W^2-5W+1)}{4}\right)$,except
$G_{Pn_1}.G_{Pn_4}+\left(\frac{n^4(3W^2-5W-1)+n^3(12W^2-2)-n^2(6W^2-5W+1)}{4}\right)$ are perfect Squares. Hence
$(G_{Pn_1},G_{Pn_2},G_{Pn_3},G_{Pn_4})$is an Irregular
rational Diophantine quadruple. □

Example:
Consider the general expression for Heptagonal and Nonagonal numbers, we get the Regular rational Diophantine quadruple triple
$\left(\frac{5n^2-3n}{2},\frac{7n^2-5n}{2},\frac{60n^4-56n^3+12n^2}{5n^2-3n}\right)$ with the property $D\left(\frac{n^4+46n^3-15n^2}{4}\right)$ and irregular rational Diophantine quadruple
$\left(\frac{5n^2-3n}{2},\frac{7n^2-5n}{2},\frac{60n^4-56n^3+12n^2}{5n^2-3n},\frac{399n^3-446n^2+115n}{2(7n-5)}\right)$ with the same property.

4|Regular rational Diophantine triples and irregular rational Diophantine quadruples with respect to Centered Pyramidal Numbers

Let $\left(CP_S = \dfrac{n(2n^2+1)}{3}, CP_O = \dfrac{n(4n^2-1)}{3}\right)$ be the centered square pyramidal and centered octagonal pyramidal Numbers,such that $CP_S CP_O + \left(\dfrac{-7n^6+2n^2}{9}\right)$ is a perfect square.We assume that CP_N is the third element in the aforementioned pair in order to check its extendability.Next,it fulfills the set of equations

$$CP_S CP_N + \left(\frac{-7n^6+2n^2}{9}\right) = A^2 \tag{11}$$

$$CP_O CP_N + \left(\frac{-7n^6+2n^2}{9}\right) = B^2 \tag{12}$$

Setting $A = X + CP_S T, B = X + CP_0 T$ and subtracting (11) from (12),we obtain $CP_N = \dfrac{16n^5+12n^3+2n}{3(2n^2+1)}$. Hence (CP_S, CP_O, CP_N) is a Regular rational Diophantine triple with the property $D\left(\dfrac{-7n^6+2n^2}{9}\right)$.

Theorem 0.4. *Let* $CP_S = \dfrac{n(2n^2+1)}{3}$,$CP_O = \dfrac{n(4n^2-1)}{3}$,$CP_N = \dfrac{16n^5+12n^3+2n}{3(2n^2+1)}$ *and* $CP_M = \dfrac{88n^5-18n^3-n}{3(4n^2-1)}$.*Then* (CP_S, CP_O, CP_N, CP_M) *is an irregular rational Diophantine quadruple with* $D\left(\dfrac{-7n^6+2n^2}{9}\right)$.

Proof: For all the combinations of the quadruple satisfying $D\left(\dfrac{-7n^6+2n^2}{9}\right)$,except $CP_S.CP_M+\left(\dfrac{-7n^6+2n^2}{9}\right)$ are perfect squares.Hence (CP_S, CP_O, CP_N, CP_M) is an irregular rational Diophantine quadruple. □

5|Conclusion

Recent research focuses on generation of Diophantine triples and quadruples from a diophantine pair with suitable property which involves polynomials.In this paper, we have come to the conclusion that not all the rational diophantine pairs and triples can be extended to rational diophantine triples and quadruples respectively.We have associated the special figurate numbers to prove the non-extendability.Researchers may search for other rational numbers to examine their extendability with suitable properties.

References

References

[1] A.Baker and H.Davenort. The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$. *Quart.J.Math.Oxford Ser*, 2(20):129–137, 1969.

[2] A.Dujella. There are only finitely many diophantine quintuples. *J.Reine Angew. Math*, 566:183–214.

[3] A.Dujella. what is a diophantine m-tuple? *Notices Amer.Math.Soc*, 63:772–774, 2016.

[4] Alan Baker. *Transcendental number theory*. Cambridge Univ. Press, Cambridge, 1975.

[5] J.Kannan and Manju Somanath. *Fundamental Perceptions in Contemporary Number Theory*. Nova Science Publisher,Inc,Ny,11788 USA, 2023.

[6] M.A.Gopalan V.sangeetha and Manju Somanath. Construction of strong and almost strong rational diophantine quadruples. *JP Journal of Algebra,Number Theory and Applications*, 35(1):35–48, 2014.

[7] Heichelheim P. The study of positive integers (a, b) such that $ab + 1$ is a square. *Fibonacci Quart*, 17:269–274, 1979.